

ON A GENERALIZATION OF COMPENSATED COMPACTNESS IN THE $L^p - L^q$ SETTING

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ABSTRACT. We investigate conditions under which, for two sequences (\mathbf{u}_r) and (\mathbf{v}_r) weakly converging to \mathbf{u} and \mathbf{v} in $L^p(\mathbf{R}^d; \mathbf{R}^N)$ and $L^q(\mathbf{R}^d; \mathbf{R}^N)$, respectively, $1/p + 1/q \leq 1$, a quadratic form $q(\mathbf{x}; \mathbf{u}_r, \mathbf{v}_r) = \sum_{j,m=1}^N q_{jm}(\mathbf{x}) u_{jr} v_{mr}$ converges toward $q(\mathbf{x}; \mathbf{u}, \mathbf{v})$ in the sense of distributions. The conditions involve fractional derivatives and variable coefficients, and they represent a generalization of the known compensated compactness theory. The proofs are accomplished using a recently introduced H -distribution concept. We apply the developed techniques to a nonlinear (degenerate) parabolic equation.

1. INTRODUCTION

The compensated compactness theory proved to be a very useful tool in investigating problems involving partial differential equations (both linear and nonlinear). Suppose, for instance, that we aim to solve a nonlinear partial differential equation which we write symbolically as $A[u] = f$, where A denotes a given nonlinear operator. One of usual approaches is to approximate it by a collection of *nicer* problems $A_r[u_r] = f_r$, where (A_r) is a sequence of operators which is somehow close to A . Then we try to prove that the sequence (u_r) converges toward a solution to the original problem $A[u] = f$. In general, it is relatively easy to obtain weak convergence on a subsequence of (u_r) towards some function u . Due to the nonlinear nature of A , this does not mean that u will represent a solution to the original problem $A[u] = f$. However, in some cases, the nonlinearity of A can be *compensated* by certain properties of the sequence (u_r) (see [3, 4, 13] and references therein). The theory which investigates such phenomena is called compensated compactness and it was introduced in the works of F. Murat and L. Tartar [14, 19].

The most general version of the classical result of compensated compactness theory has been recently proved in [15]. Let us briefly recall it. First, we introduce anisotropic Sobolev spaces $W^{-1,-2;p}(\mathbf{R}^d)$, where -1 is with respect to x_1, \dots, x_ν and -2 is with respect to $x_{\nu+1}, \dots, x_d$, as a subset of tempered distributions

$$\{u \in \mathcal{S}' : \exists v \in L^p(\mathbf{R}^d), k\hat{u} = \hat{v}\},$$

where $k(\xi_1, \xi_2) = \sqrt{1 + (2\pi|\xi_1|)^2 + (2\pi|\xi_2|)^4}$, $\xi_1 \in \mathbf{R}^\nu$, $\xi_2 \in \mathbf{R}^{d-\nu}$. It is Hörmander's class $B_{p,k}$ and the Banach space with dual $B_{p',1/k}$ (see chapter 10 of [9]). By \hat{u} we denote the Fourier transform: $\hat{u}(\xi) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x} \cdot \xi} u(\mathbf{x}) d\mathbf{x}$.

1991 *Mathematics Subject Classification.* 35A27, 35K55, 46G10, 42B15, 42B30.

Key words and phrases. $L^p - L^q$ compensated compactness; H -distributions; non-strictly parabolic equations; weak convergence method.

Assume that the sequence $(\mathbf{u}_r) = (u_{1r}, \dots, u_{Nr})$ is bounded in $L^p(\mathbf{R}^d; \mathbf{R}^N)$, $2 \leq p \leq \infty$, and converges in $\mathcal{D}'(\mathbf{R}^d)$ to a vector function \mathbf{u} . Let $q = \frac{p}{p-1}$ if $p < \infty$, and $q > 1$ if $p = \infty$. Assume that the sequences

$$\sum_{j=1}^N \sum_{k=1}^{\nu} \partial_{x_k} (a_{sjk} u_{jr}) + \sum_{j=1}^N \sum_{k,l=\nu+1}^d \partial_{x_k x_l} (b_{sjkl} u_{jr}), \quad (1)$$

for $s = 1, \dots, m$, are precompact in the anisotropic Sobolev space $W_{loc}^{-1,-2;q}(\mathbf{R}^d)$. The (variable) coefficients a_{sjk} and b_{sjkl} belong to $L^{2\bar{q}}(\mathbf{R}^d)$, $\bar{q} = \frac{p}{p-2}$ if $p > 2$, and to the space $C(\mathbf{R}^d)$ if $p = 2$.

Next, introduce the set

$$\Lambda(\mathbf{x}) = \left\{ \boldsymbol{\lambda} \in \mathbf{R}^N \mid (\exists \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{0\})(\forall s = 1, \dots, m) \right. \\ \left. \sum_{j=1}^N \left(i \sum_{k=1}^{\nu} a_{sjk}(\mathbf{x}) \xi_k - \sum_{k,l=\nu+1}^d b_{sjkl}(\mathbf{x}) \xi_k \xi_l \right) \lambda_j = 0 \right\}. \quad (2)$$

Consider the bilinear form on \mathbf{R}^N

$$q(\mathbf{x}; \boldsymbol{\lambda}, \boldsymbol{\eta}) = Q(\mathbf{x}) \boldsymbol{\lambda} \cdot \boldsymbol{\eta}, \quad (3)$$

where Q is a symmetric matrix with coefficients

$$q_{jm} \in \begin{cases} L_{loc}^{\bar{q}}(\mathbf{R}^d), & p > 2 \\ C(\mathbf{R}^d), & p = 2 \end{cases}, \quad j, m = 1, \dots, N.$$

Finally, let $q(\mathbf{x}; \mathbf{u}_r, \mathbf{u}_r) \rightharpoonup \omega$ weakly-* in the space of Radon measures.

The following theorem holds

Theorem 1. [15, Theorem 1] *Assume that $q(\mathbf{x}; \boldsymbol{\lambda}, \boldsymbol{\lambda}) \geq 0$ for all $\boldsymbol{\lambda} \in \Lambda(\mathbf{x})$, a.e. $\mathbf{x} \in \mathbf{R}^d$. Then $q(\mathbf{x}; \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{x})) \leq \omega$ in the sense of measures. If $q(\mathbf{x}; \boldsymbol{\lambda}, \boldsymbol{\lambda}) = 0$ for all $\boldsymbol{\lambda} \in \Lambda(\mathbf{x})$, a.e. $\mathbf{x} \in \mathbf{R}^d$, then $q(\mathbf{x}; \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{x})) = \omega$.*

The connection between q and Λ given in the previous theorem, we shall call *the consistency condition*.

We would like to formulate and extend the results from Theorem 1 to the $L^p - L^q$ framework for appropriate (greater than one) indices p and q where $p < 2$. More precisely, we want to find conditions on two vector-valued sequences (\mathbf{u}_r) and (\mathbf{v}_r) weakly converging to \mathbf{u} and \mathbf{v} in $L^p(\mathbf{R}^d)$ and $L^q(\mathbf{R}^d)$, respectively, to ensure that the sequence $(q(\mathbf{x}; \mathbf{u}_r, \mathbf{v}_r))$, where q is the bilinear form from (3), satisfies

$$\lim_{r \rightarrow \infty} q(\mathbf{x}; \mathbf{u}_r, \mathbf{v}_r) = q(\mathbf{x}; \mathbf{u}, \mathbf{v}) \quad \text{in } \mathcal{D}'(\mathbf{R}^d). \quad (4)$$

Ideally, it should be $1/p + 1/q = 1$. Due to technical obstacles (see Remark 10), we are able to prove (4) only when $1/p + 1/q < 1$. However, under additional assumptions on the sequences (\mathbf{u}_r) and (\mathbf{v}_r) , we are also able to obtain the optimal $L^p - L^{p'}$ -variant of the compensated compactness. Here and in the sequel, $1/p + 1/p' = 1$.

This extension will be done in the next section. In the last section we shall show how to apply this result to a (nonlinear) parabolic type equation.

2. THE MAIN RESULT

In order to formulate the $L^p - L^q$ variant of the compensated compactness, we need H -distributions.

They were introduced in [3] as an extension of the H -measure concept (see [7, 19, 2, 10] and references therein). Let us recall that H -measures describe the loss of strong precompactness for sequences belonging to L^p for $p \geq 2$, and they were the basic tool in the mentioned work on compensated compactness [15]. The variant of H -distributions that we are basically going to use is formulated in [11, 12]. Let us recall its definition.

We need multiplier operators with symbols defined on a manifold P determined by an d -tuple $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{R}_+^d$, where $\alpha_k \in \mathbf{N}$ or $\alpha_k \geq d$

$$P = \left\{ \xi \in \mathbf{R}^d : \sum_{k=1}^d |\xi_k|^{2\alpha_k} = 1 \right\}.$$

The manifold P is smooth enough and we are able to associate an L^p multiplier to a function defined on P as follows. We define the projection from $\mathbf{R}^d \setminus \{0\}$ to P by means of the mapping

$$(\pi_P(\xi))_j = \xi_j \left(|\xi_1|^{2\alpha_1} + \dots + |\xi_d|^{2\alpha_d} \right)^{-1/2\alpha_j}, \quad j = 1, \dots, d, \quad \xi \in \mathbf{R}^d \setminus \{0\}.$$

Let us now recall the Marcinkiewicz multiplier theorem [8, Theorem 5.2.4.], more precisely its corollary which we provide here:

Corollary 2. *Suppose that $\psi \in C^d(\mathbf{R}^d \setminus \cup_{j=1}^d \{\xi_j = 0\})$ is a bounded function such that for some constant $C > 0$ it holds*

$$|\xi^{\tilde{\alpha}} \partial^{\tilde{\alpha}} \psi(\xi)| \leq C, \quad \xi \in \mathbf{R}^d \setminus \cup_{j=1}^d \{\xi_j = 0\} \quad (5)$$

for every multi-index $\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_d) \in \mathbf{N}_0^d$ such that $|\tilde{\alpha}| = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \dots + \tilde{\alpha}_d \leq d$. Then, the function ψ is an L^p -multiplier for $p \in \langle 1, \infty \rangle$, and the operator norm of \mathcal{A}_ψ equals to $C_{d,p}$, where $C_{d,p}$ depends only on C , p and d .

The following statement holds.

Theorem 3. [11] *Let (u_n) be a bounded sequence in $L^p(\mathbf{R}^d)$, $p > 1$, and let (v_n) be a bounded sequence of uniformly compactly supported functions in $L^\infty(\mathbf{R}^d)$ weakly converging to 0 in the sense of distributions. Then, after passing to a subsequence (not relabelled), for any $\bar{p} \in \langle 1, p \rangle$ there exists a continuous bilinear functional B on $L^{\bar{p}'}(\mathbf{R}^d) \otimes C^d(P)$ such that for every $\varphi \in L^{\bar{p}'}(\mathbf{R}^d)$ and $\psi \in C^d(P)$ it holds*

$$B(\varphi, \psi) = \lim_{n \rightarrow \infty} \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_n(\mathbf{x}) \overline{(\mathcal{A}_{\psi_P} v_n)(\mathbf{x})} d\mathbf{x}, \quad (6)$$

where \mathcal{A}_{ψ_P} is the (Fourier) multiplier operator on \mathbf{R}^d associated to $\psi \circ \pi_P$ and $\frac{1}{p} + \frac{1}{\bar{p}'} = 1$.

The bound of the functional B is equal to $C_u C_v C_{d,q}$, where C_u is the L^p -bound of the sequence (u_n) ; C_v is the L^q -bound of the sequence (v_n) where $\frac{1}{p} + \frac{1}{\bar{p}'} + \frac{1}{q} = 1$; and $C_{d,q}$ is the constant from Corollary 2.

We shall now prove that we can extend the bilinear functional B from the previous theorem to a functional on $L^{\bar{p}'}(\mathbf{R}^d; C^d(P))$. We shall need the following theorem a proof of which in the case of real functionals can be found in [11].

Theorem 4. *Let B be a (complex valued) continuous bilinear functional on $L^p(\mathbf{R}^d) \otimes E$, where E is a separable Banach space and $p \in \langle 1, \infty \rangle$. Then B can be extended as a (complex valued) continuous functional on $L^p(\mathbf{R}^d; E)$ if and only if there exists a (nonnegative) function $b \in L^{p'}(\mathbf{R}^d)$ such that for every $\psi \in E$ and almost every $\mathbf{x} \in \mathbf{R}^d$, it holds*

$$|\tilde{B}\psi(\mathbf{x})| \leq b(\mathbf{x})\|\psi\|_E, \quad (7)$$

where \tilde{B} is a bounded linear operator $E \rightarrow L^{p'}(\mathbf{R}^d)$ defined by $\langle \tilde{B}\psi, \varphi \rangle = B(\varphi, \psi)$, $\varphi \in L^p(\mathbf{R}^d)$.

Proof: The proof goes along the lines of the proof of [11, Theorem 2.1] when we separately consider real (\Re) and imaginary (\Im) parts of the functional B and the operator \tilde{B} . Let us briefly recall it.

Let us assume that (7) holds. In order to prove that B can be extended as a linear functional on $L^p(\mathbf{R}^d; E)$, it is enough to obtain an appropriate bound on the following dense subspace of $L^p(\mathbf{R}^d; E)$:

$$\left\{ \sum_{j=1}^N \psi_j \chi_j(\mathbf{x}) : \psi_j \in E, N \in \mathbf{N} \right\}, \quad (8)$$

where χ_i are characteristic functions associated to mutually disjoint, finite measure sets.

For an arbitrary function $g = \sum_{i=1}^N \psi_i \chi_i$ from (8), the bound follows easily once we notice that

$$\begin{aligned} \left| B\left(\sum_{j=1}^N \psi_j \chi_j\right) \right| &:= \left| \sum_{j=1}^N B(\chi_j, \psi_j) \right| = \left| \int_{\mathbf{R}^d} \sum_{j=1}^N \tilde{B}\psi_j(\mathbf{x}) \chi_j(\mathbf{x}) d\mathbf{x} \right| \\ &\leq \int_{\mathbf{R}^d} b(\mathbf{x}) \sum_{j=1}^N \chi_j(\mathbf{x}) \|\psi_j\|_E d\mathbf{x} \leq \|b\|_{L^{p'}(\mathbf{R}^d)} \|g\|_{L^p(\mathbf{R}^d; E)}. \end{aligned}$$

In order to prove the converse, take a countable dense set of functions from the unit ball of E , and denote them by ψ_j , $j \in \mathbf{N}$. Assume that the functions $\psi_{-j} := -\psi_j$ are also in E . For each function $\tilde{B}\psi_j \in L^{p'}(\mathbf{R}^d)$ denote by D_j the corresponding set of Lebesgue points, and their intersection by $D = \cap_j D_j$.

For any $\mathbf{x} \in D$ and $k \in \mathbf{N}$ denote

$$b_k(\mathbf{x}) = \max_{|j| \leq k} \Re(\tilde{B}\psi_j)(\mathbf{x}) = \sum_{|j|=1}^k \Re(\tilde{B}\psi_j)(\mathbf{x}) \chi_j^k(\mathbf{x})$$

where $\chi_{j_0}^k$ is the characteristic function of set $X_{j_0}^k$ of all points $\mathbf{x} \in D$ for which the above maximum is achieved for $j = j_0$. Furthermore, we can assume that for each k the sets X_j^k are mutually disjoint. The sequence (b_k) is clearly monotonic sequence of positive functions, bounded in $L^{p'}(\mathbf{R}^d)$, whose limit (in the same space)

we denote by b^\Re . Indeed, choose $\varphi \in L^p(\mathbf{R}^d)$, $g = \sum_{|j|=1}^k \varphi(\mathbf{x}) \chi_j^k(\mathbf{x}) \psi_j \in L^p(\mathbf{R}^d; E)$,

and consider:

$$\begin{aligned} \int_{\mathbf{R}^d} b_k(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} &= \Re\left(\int_{\mathbf{R}^d} \tilde{B} \sum_{|j|=1}^k \psi_j \chi_j^k(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}\right) \\ &= \Re\left(\sum_{|j|=1}^k B(\chi_j^k \varphi, \psi_j)\right) = \Re(B(g)) \leq C\|g\|_{L^p(\mathbf{R}^d; E)} \leq C\|\varphi\|_{L^p(\mathbf{R}^d)}, \end{aligned}$$

where C is the norm of B on $(L^p(\mathbf{R}^d; E))'$. Since $\varphi \in L^p(\mathbf{R}^d)$ is arbitrary, we get that (b_k) is bounded in $L^{p'}(\mathbf{R}^d)$.

As D is a set of full measure, for every ψ_j we have

$$|\Re(\tilde{B}\psi_j)(\mathbf{x})| \leq b^{\Re}(\mathbf{x}), \quad (\text{a.e. } \mathbf{x} \in \mathbf{R}^d).$$

We are able to obtain a similar bound for the imaginary part of $\tilde{B}\psi_j$. In other words, there exists $b^{\Im} \in L^{p'}(\mathbf{R}^d)$ such that

$$|\Im(\tilde{B}\psi_j)(\mathbf{x})| \leq b^{\Im}(\mathbf{x}), \quad (\text{a.e. } \mathbf{x} \in \mathbf{R}^d).$$

The assertion now follows since (7) holds for $b = b^{\Re} + b^{\Im}$ on the dense set of functions ψ_j , $j \in \mathbf{N}$. For details see (12) below. \square

We need the following lemma which will also be used in the last section.

Lemma 5. *If the real symbol $\psi \in C^d(\mathbf{P})$ of the multiplier operator \mathcal{A}_ψ is an even function ($\psi(\xi) = \psi(-\xi)$), then for every real $u \in L^p(\mathbf{R}^d)$, $p > 1$, $\mathcal{A}_\psi(u)$ is a real function for a.e. $\mathbf{x} \in \mathbf{R}^d$.*

If the real symbol $\psi \in C^d(\mathbf{P})$ of the multiplier operator \mathcal{A}_ψ is an odd function ($\psi(\xi) = -\psi(-\xi)$), then for every real $u \in L^p(\mathbf{R}^d)$, $p > 1$, $\mathcal{A}_\psi(u)$ is a purely imaginary function for a.e. $\mathbf{x} \in \mathbf{R}^d$.

Proof: Assume first that the symbol ψ is an even function. It is enough to prove that, for arbitrary real functions $u, v \in L^2(\mathbf{R}^d) \cap L^p(\mathbf{R}^d)$, it holds

$$\int v \mathcal{A}_\psi(u) d\mathbf{x} = \int v \overline{\mathcal{A}_\psi(u)} d\mathbf{x}.$$

This follows from the Plancherel theorem, and the change of variables $\xi \mapsto -\xi$. Indeed,

$$\begin{aligned} \int v \mathcal{A}_\psi(u) d\mathbf{x} &= \int \bar{v} \mathcal{A}_\psi(u) d\mathbf{x} = \int \psi(\xi) \bar{\hat{v}}(\xi) \hat{u}(\xi) d\xi = (\xi \mapsto -\xi) \\ &= \int \psi(\xi) \hat{v}(\xi) \bar{\hat{u}}(\xi) d\xi = \int v \overline{\mathcal{A}_\psi(u)} d\mathbf{x}. \end{aligned}$$

The proof is the same when the symbol is odd. \square

Now, we can prove the following proposition.

Proposition 6. [11] *The bilinear functional B defined in Theorem 3 can be extended by continuity to a functional on $L^{\bar{p}'}(\mathbf{R}^d; C^d(\mathbf{P}))$. The bound of the extension is equal to the bound of the bilinear functional B (with the notations of Theorem 3, it is $C_u C_v C_{d,q}$, $1/p + 1/\bar{p}' + 1/q = 1$).*

Remark 7. The proof of the proposition can also be found in [12]. Since this paper is still unpublished, we give a slightly different proof here.

Proof: We will show that B satisfies conditions of Theorem 4, namely, that there exists a function $b \in L^{\bar{p}}(\mathbf{R}^d)$ such that for every $\psi \in C^d(\mathbf{P})$, $\|\psi\|_{C^d(\mathbf{P})} \leq 1$ and almost every $\mathbf{x} \in \mathbf{R}^d$ it holds

$$|(\tilde{B}\psi)(\mathbf{x})| \leq b(\mathbf{x})\|\psi\|_{C^d(\mathbf{P})}, \quad (9)$$

where $\tilde{B} : C^d(\mathbf{P}) \rightarrow L^{\bar{p}}(\mathbf{R}^d)$ is a bounded linear operator defined by $\langle \tilde{B}\psi, \varphi \rangle = B(\varphi, \psi)$, $\varphi \in L^{\bar{p}'}(\mathbf{R}^d)$.

We proceed as follows: choose a dense countable set E of functions ψ_j , $j \in \mathbf{N}$, from the set $\{\psi \in C^d(\mathbf{P}) : \|\psi\|_{C^d(\mathbf{P})} \leq 1\}$. Define functions $\psi_{-j}(\xi) = -\psi_j(\xi)$ and add them to E . Moreover, add the linear combinations of the form $\psi_j^e(\xi) = \frac{1}{2}(\psi_j(\xi) + \psi_j(-\xi))$ and $\psi_j^o(\xi) = \frac{1}{2}(\psi_j(\xi) - \psi_j(-\xi))$ for $j \in \mathbf{Z} \setminus \{0\}$ to E as well. Remark that functions ψ_j^e are even, while ψ_j^o are odd (in the sense of Lemma 5) and that the set E is still countable and dense.

For each j choose a function $\tilde{B}\psi_j$ from $L^{\bar{p}}(\mathbf{R}^d)$ and denote by D_j the corresponding set of Lebesgue points (for definiteness, we can take $\tilde{B}\psi_j$ to be the precise representative of the class (see chapter 1.7. of [6])). The set D_j is of full measure, and thus the set $D = \cap_j D_j$ as well.

For any $\mathbf{x} \in D$ and $k \in \mathbf{N}$ denote ($i = \sqrt{-1}$ below)

$$b_k^e(\mathbf{x}) := \max_{|j| \leq k} \tilde{B}\psi_j^e(\mathbf{x}) = \sum_{|j|=1}^k \tilde{B}\psi_j^e(\mathbf{x})\chi_j^k(\mathbf{x}) \in \mathbf{R}^+, \quad (10)$$

$$b_k^o(\mathbf{x}) := \max_{|j| \leq k} i\tilde{B}\psi_j^o(\mathbf{x}) = \sum_{|j|=1}^k i\tilde{B}\psi_j^o(\mathbf{x})\chi_j^k(\mathbf{x}) \in \mathbf{R}^+, \quad (11)$$

where $\chi_{j_0}^k$ is a characteristic function of the set of all points for which the above maximum is achieved for $\psi_{j_0}^e$ ($\psi_{j_0}^o$ respectively) and it has not been achieved for ψ_j^e (ψ_j^o respectively), $-k \leq j < j_0$.

First, note that we can make sure that χ_j^k have disjoint supports for fixed k : define χ_j^k to be equal to one on the set

$$\left\{ \mathbf{x} \in D : (\tilde{B}\psi_j^e)(\mathbf{x}) = b_k^e(\mathbf{x}) \text{ \& } (\forall l < j)(\tilde{B}\psi_l^e)(\mathbf{x}) < b_k^e(\mathbf{x}) \right\},$$

and extend it with zero to the whole \mathbf{R}^d .

Next, we shall prove that the sequence of functions (b_k^e) is bounded in $L^{\bar{p}}(\mathbf{R}^d)$. To this effect, take an arbitrary $\phi \in C_c(\mathbf{R}^d)$, and denote $K = \text{supp } \phi$. Since (v_n) is a bounded sequence of uniformly compactly supported functions in $L^\infty(\mathbf{R}^d)$, it belongs to $L^q(\mathbf{R}^d)$ for every $q \in \langle 1, \infty \rangle$. Since $\bar{p} < p$, we can find $q > 1$ such that $1/q + 1/\bar{p}' = 1/p'$. Fix such q . Choose $r > 1$ such that $q = r'p'$. Denote by $\chi_j^{k,\varepsilon} \in C_c(\mathbf{R}^d)$, $j = 1, \dots, k$ smooth approximations of characteristic functions from (10) on K such that (note that $\|\chi_j^k\|_{L^\infty} \leq 1$)

$$\|\chi_j^{k,\varepsilon} - \chi_j^k\|_{L^{\max\{p', r\}}(K)} \leq \frac{\varepsilon}{2k}.$$

As before, denote by C_u an L^p bound of (u_n) and by C_v an L^q bound of (v_n) .

According to (10) and the definition of operator \tilde{B} , we have

$$\begin{aligned}
& \left| {}_{L^{\bar{p}}(\mathbf{R}^d)} \langle b_k^e, \phi \rangle_{{}_{L^{\bar{p}'}}(\mathbf{R}^d)} \right| = \left| \lim_{n \rightarrow \infty} \int_{\mathbf{R}^d} \sum_{|j|=1}^k (\phi u_n \chi_j^k)(\mathbf{x}) (\overline{\mathcal{A}_{\psi_j^e} v_n})(\mathbf{x}) d\mathbf{x} \right| \\
& \leq \limsup_{n \rightarrow \infty} \int_{\mathbf{R}^d} \left(\sum_{|j|=1}^k |u_n|^p \chi_j^k(\mathbf{x}) \right)^{1/p} \left(\sum_{|j|=1}^k \chi_j^k |\phi \mathcal{A}_{\psi_j^e} v_n|^{p'}(\mathbf{x}) \right)^{1/p'} d\mathbf{x} \\
& \leq \limsup_{n \rightarrow \infty} \left\| \sum_{|j|=1}^k |u_n|^p \chi_j^k \right\|_{L^1(\mathbf{R}^d)}^{1/p} \left\| \sum_{|j|=1}^k \chi_j^k |\phi \mathcal{A}_{\psi_j^e} v_n|^{p'} \right\|_{L^1(\mathbf{R}^d)}^{1/p'} \\
& \leq \limsup_{n \rightarrow \infty} \|u_n\|_{L^p(\mathbf{R}^d)} \left(\left\| \sum_{|j|=1}^k (\chi_j^k - \chi_j^{k,\varepsilon}) |\phi \mathcal{A}_{\psi_j^e} v_n|^{p'} \right\|_{L^1(\mathbf{R}^d)} \right. \\
& \quad \left. + \sum_{|j|=1}^k \left\| \chi_j^{k,\varepsilon} |\phi \mathcal{A}_{\psi_j^e} v_n|^{p'} \right\|_{L^1(\mathbf{R}^d)} \right)^{1/p'} \\
& \leq \limsup_{n \rightarrow \infty} \|u_n\|_{L^p(\mathbf{R}^d)} \left(\left\| \sum_{|j|=1}^k (\chi_j^k - \chi_j^{k,\varepsilon}) |\phi \mathcal{A}_{\psi_j^e} v_n|^{p'} \right\|_{L^1(\mathbf{R}^d)} \right. \\
& \quad \left. + \sum_{|j|=1}^k \left\| \chi_j^{k,\varepsilon} |\phi \mathcal{A}_{\psi_j^e} v_n|^{p'} \right\|_{L^{p'}(\mathbf{R}^d)} \right)^{1/p'} \\
& \leq C_u \limsup_{n \rightarrow \infty} \left(\sum_{|j|=1}^k \|\chi_j^k - \chi_j^{k,\varepsilon}\|_{L^r(K)} \|\mathcal{A}_{\psi_j^e}(\phi v_n)\|_{L^q(\mathbf{R}^d)}^{p'} \right. \\
& \quad \left. + \sum_{|j|=1}^k \|\mathcal{A}_{\psi_j^e}(\chi_j^{k,\varepsilon} \phi v_n)\|_{L^{p'}(\mathbf{R}^d)}^{p'} \right)^{1/p'},
\end{aligned}$$

where in the second step we have used discrete version of Hölder inequality and the fact that $|\lim_n a_n| \leq \limsup_n |a_n|$; in the last step we have used a version of the first commutation lemma [3, Lemma 3.1] (see also [12, Lemma 2]) and Hölder inequality with $1/r + 1/r' = 1$ remembering that $r'p' = q$. By means of Corollary 2 and properties of the functions $\chi_j^{k,\varepsilon}$ it follows

$$\left| \langle b_k^e, \phi \rangle \right| \leq C_u \limsup_{n \rightarrow \infty} \left(\varepsilon C_\phi C_{q,d}^{p'} \|v_n\|_{L^q(\mathbf{R}^d)}^{p'} + C_{p',d}^{p'} \sum_{|j|=1}^k \|\chi_j^{k,\varepsilon} \phi v_n\|_{L^{p'}(\mathbf{R}^d)}^{p'} \right)^{1/p'},$$

where $C_{p',d}$ is the constant from Corollary 2 (recall that $\|\psi_j^e\|_{C^d(\mathbf{P})} \leq 1$), while $C_\phi = \|\phi\|_{L^\infty(\mathbf{R}^d)}^{p'}$. By letting $\varepsilon \rightarrow 0$, we conclude

$$\left| \langle b_k^e, \phi \rangle \right| \leq C_u C_{p',d} \limsup_{n \rightarrow \infty} \left(\sum_{|j|=1}^k \|\chi_j^k \phi v_n\|_{L^{p'}(\mathbf{R}^d)}^{p'} \right)^{1/p'}$$

since $\chi_j^{k,\varepsilon} \rightarrow \chi_j^k$ in $L^{p'}(K)$. Since supports of functions χ_j^k are disjoint and remembering the choice of q , we get

$$\sum_{|j|=1}^k \|\chi_j^k \phi v_n\|_{L^{p'}(\mathbf{R}^d)}^{p'} \leq \|\phi v_n\|_{L^{p'}(\mathbf{R}^d)}^{p'} \leq \left(\|\phi\|_{L^{\bar{p}}(\mathbf{R}^d)} \|v_n\|_{L^q(\mathbf{R}^d)} \right)^{p'},$$

since $\sum_{|j|=1}^k (\chi_j^k)^{p'} = \sum_{|j|=1}^k \chi_j^k \leq 1$. From this, it follows

$$|\langle b_k^e, \phi \rangle| \leq C_u C_{d,p'} C_v \|\phi\|_{L^{\bar{p}}(\mathbf{R}^d)},$$

where all the constants on the right hand side do not depend on k . Since $C_c(\mathbf{R}^d)$ is dense in $L^{\bar{p}}(\mathbf{R}^d)$ we conclude that the sequence (b_k^e) is bounded in $L^{\bar{p}}(\mathbf{R}^d)$. Noticing that (b_k^e) is a non-decreasing sequence of positive functions, it follows from Beppo-Levi's theorem on monotone convergence that its (pointwise) limit b^e is an $L^{\bar{p}}(\mathbf{R}^d)$ function.

In the completely same way, we conclude that (b_k^o) converges toward $b^o \in L^{\bar{p}}(\mathbf{R}^d)$.

The function $b = b^e + b^o$ satisfies (9) for $\tilde{B}\psi$ when $\psi = \psi_j^e + \psi_{j'}^o$, for some $j, j' \in \mathbf{Z} \setminus \{0\}$. On the other hand, every $\psi \in C^d(\mathbf{P})$ can be represented as a sum of odd and even functions as follows $\psi(\mathbf{x}) = \frac{1}{2}(\psi(\mathbf{x}) + \psi(-\mathbf{x})) + \frac{1}{2}(\psi(\mathbf{x}) - \psi(-\mathbf{x}))$ and we conclude that (9) holds for any $\psi \in E$. By continuity, the statement can be generalised to an arbitrary $\psi \in C^d(\mathbf{P})$: take a sequence $(\psi_n) \subseteq E$ such that $\psi_n \rightarrow \psi$ in $C^d(\mathbf{P})$ and write

$$\begin{aligned} \int_{\mathbf{R}^d} |(\tilde{B}\psi)(\mathbf{x})| \varphi(\mathbf{x}) d\mathbf{x} &\leq \int_{\mathbf{R}^d} |(\tilde{B}\psi - \tilde{B}\psi_n)(\mathbf{x})| \varphi(\mathbf{x}) d\mathbf{x} + \int_{\mathbf{R}^d} |(\tilde{B}\psi_n)(\mathbf{x})| \varphi(\mathbf{x}) d\mathbf{x} \\ &\leq o_n(1) + \int_{\mathbf{R}^d} b(\mathbf{x}) \varphi(\mathbf{x}) d\mathbf{x}, \end{aligned} \tag{12}$$

for arbitrary $\varphi \in C_c^\infty(\mathbf{R}^d; \mathbf{R}_0^+)$ where we have used continuity of \tilde{B} . Due to arbitrariness of the function φ , the result follows from Theorem 4. \square

Remark 8. Note that if the set $L := \{\psi \in C^d(\mathbf{P}) : \|\psi\|_{C^d(\mathbf{P})} \leq 1\}$ were at most countable, we could have defined $b \in L^{\bar{p}}(\mathbf{R}^d)$ in the following straightforward way

$$b(\mathbf{x}) = \sup_{\psi \in L} |(\tilde{B}\psi)(\mathbf{x})|.$$

However, L is uncountable, so this definition does not necessarily result in a measurable function. Taking supremum over a countable dense subset of L would result in a measurable function which may not be $L^{\bar{p}}$ -function.

Now, we are ready to prove a variant of compensated compactness in the $L^p - L^q$ framework. Before we proceed, we recall that the dual of the space $L^p(\mathbf{R}^d; C^d(\mathbf{P}))$ is the space $L_{w*}^{p'}(\mathbf{R}^d; C^d(\mathbf{P})')$ of weakly-* measurable functions $B : \mathbf{R}^d \rightarrow C^d(\mathbf{P})'$ such that $\int_{\mathbf{R}^d} \|B(\mathbf{x})\|_{C^d(\mathbf{P})'}^{p'} d\mathbf{x}$ is finite (for details see [5, p. 606]).

We first need to extend the notion of H -distributions from Theorem 3 as follows.

Theorem 9. *Let (u_r) be a sequence of uniformly compactly supported functions weakly converging to zero in $L^p(\mathbf{R}^d)$, $p > 1$, and let (v_r) be a bounded sequence of uniformly compactly supported functions in $L^q(\mathbf{R}^d)$, $1/q + 1/p < 1$, weakly converging to 0 in the sense of distributions. Then, after passing to a subsequence (not*

relabelled), for any $\bar{p} \in \langle 1, \frac{pq}{p+q} \rangle$ there exists a continuous bilinear functional B on $L^{\bar{p}'}(\mathbf{R}^d) \otimes C^d(\mathbf{P})$ such that for every $\varphi \in L^{\bar{p}'}(\mathbf{R}^d)$ and $\psi \in C^d(\mathbf{P})$, it holds

$$B(\varphi, \psi) = \lim_{r \rightarrow \infty} \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_r(\mathbf{x}) \overline{(\mathcal{A}_{\psi_{\mathbf{P}}} v_r)(\mathbf{x})} d\mathbf{x}, \quad (13)$$

where $\mathcal{A}_{\psi_{\mathbf{P}}}$ is the (Fourier) multiplier operator on \mathbf{R}^d associated to $\psi \circ \pi_{\mathbf{P}}$.

The bilinear functional B can be continuously extended to a linear functional on $L^{\bar{p}'}(\mathbf{R}^d; C^d(\mathbf{P}))$.

Proof: Introduce the truncation operator

$$T_l(v) = \begin{cases} v, & |v| < l \\ 0, & |v| \geq l \end{cases}, \quad l \in \mathbf{N}, \quad (14)$$

and rewrite v_r in the form

$$v_r(\mathbf{x}) = T_l(v_r)(\mathbf{x}) + (v_r - T_l(v_r))(\mathbf{x}),$$

where $T_l(v_r)$ is understood pointwisely. Notice that

$$\limsup_{l, r \rightarrow \infty} \|v_r - T_l(v_r)\|_{L^1(K)} = 0 \quad (15)$$

for any relatively compact measurable $K \subseteq \mathbf{R}^d$. Indeed, denote by

$$\Omega_r^l = \{\mathbf{x} \in \mathbf{R}^d : |v_r(\mathbf{x})| > l\}.$$

It holds

$$\lim_{l \rightarrow \infty} \sup_{r \in \mathbf{N}} \text{meas}(\Omega_r^l) = 0. \quad (16)$$

The latter follows since (v_r) is bounded in $L^q(\mathbf{R}^d)$ and

$$\sup_{r \in \mathbf{N}} \int_{\mathbf{R}^d} |v_r(\mathbf{x})|^q d\mathbf{x} \geq \sup_{r \in \mathbf{N}} \int_{\Omega_r^l} l^q d\mathbf{x} \geq l^q \sup_{r \in \mathbf{N}} \text{meas}(\Omega_r^l).$$

Now, we simply use the Hölder inequality

$$\int_K |v_r - T_l(v_r)| dx = \int_{K \cap \Omega_r^l} |v_r| dx \leq \text{meas}(K \cap \Omega_r^l)^{1/q'} \|v_r\|_{L^q(K)}$$

and this tends to zero uniformly with respect to r and l according to (16) and the boundedness of (v_r) in $L^q(\mathbf{R}^d)$. Thus, (15) is proved. Since (v_r) , and therefore $(T_l(v_r))$ are bounded in $L^q(\mathbf{R}^d)$, (15) and interpolation inequalities imply that for any $\bar{q} \in [1, q]$

$$\limsup_{l, r \rightarrow \infty} \|v_r - T_l(v_r)\|_{L^{\bar{q}}(K)} = 0. \quad (17)$$

Next, denote by μ_l the H -distribution corresponding to (u_r) and $(T_l(v_r))$ in the sense of Theorem 3. From here and (15), we conclude that we can rewrite the right-hand side of (13) in the form

$$\begin{aligned} & \lim_{r \rightarrow \infty} \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_r(\mathbf{x}) \overline{(\mathcal{A}_{\psi_{\mathbf{P}}} v_r)(\mathbf{x})} d\mathbf{x} \\ &= \lim_{r \rightarrow \infty} \left(\int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_r(\mathbf{x}) \overline{(\mathcal{A}_{\psi_{\mathbf{P}}} (T_l(v_r)))(\mathbf{x})} d\mathbf{x} + \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_r(\mathbf{x}) \overline{(\mathcal{A}_{\psi_{\mathbf{P}}} (v_r - T_l(v_r)))(\mathbf{x})} d\mathbf{x} \right) \\ &= \langle \mu_l, \varphi \psi \rangle + o_l(1), \end{aligned} \quad (18)$$

where $\alpha_l(1) \rightarrow 0$ as $l \rightarrow \infty$ follows from (17) and the application of the Hölder inequality as follows:

$$\begin{aligned} & \left| \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_r(\mathbf{x}) \overline{\mathcal{A}_{\psi_P}(v_r - T_l(v_r))}(\mathbf{x}) d\mathbf{x} \right| \\ & \leq C_{d,\bar{q}} \|\varphi\|_{L^{\bar{p}'}(\mathbf{R}^d)} \|\psi\|_{C^d(P)} \sup_r \|u_r\|_{L^p(\mathbf{R}^d)} \sup_r \|v_r - T_l(v_r)\|_{L^{\bar{q}}(\mathbf{R}^d)}, \end{aligned}$$

where $1/\bar{p}' + 1/p + 1/\bar{q} = 1$ (and obviously $\bar{q} < q$ implying that we can apply (17)).

Since $\psi \circ \pi_P$ is an $L^{\bar{q}}$ -multiplier ([10, Lemma 5]), by the Hölder inequality used with the exponents \bar{p}' , p , and $\bar{q} < q$, we get

$$\begin{aligned} \left| \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_r(\mathbf{x}) \overline{(\mathcal{A}_{\psi_P} T_l(v_r))}(\mathbf{x}) d\mathbf{x} \right| & \leq C_{d,\bar{q}} \|\varphi\|_{L^{\bar{p}'}(\mathbf{R}^d)} \|u_r\|_{L^p(\mathbf{R}^d)} \|\psi\|_{C^d(P)} \|T_l(v_r)\|_{L^{\bar{q}}(\mathbf{R}^d)} \\ & \leq C_u C_v C_{d,\bar{q}} \|\varphi\|_{L^{\bar{p}'}(\mathbf{R}^d)} \|\psi\|_{C^d(P)} \end{aligned}$$

From here, after passing to the limit $r \rightarrow \infty$ and using the continuity of extension from Proposition 6, we conclude that (μ_l) is bounded sequence in $(L^{\bar{p}'}(\mathbf{R}^d; C^d(P)))' = L_{w*}^{\bar{p}}(\mathbf{R}^d; C^d(P)')$ (remark that the bound of (μ_l) is $C_u C_v C_{d,\bar{q}}$). Since $L_{w*}^{\bar{p}}(\mathbf{R}^d; C^d(P)')$ is dual of the Banach space, according to the Banach-Alaoglu theorem, (μ_l) admits a weak-* limit $\mu \in L_{w*}^{\bar{p}}(\mathbf{R}^d; C^d(P)')$ along a subsequence. The functional μ satisfies (13). \square

Remark 10. In the case $1/p + 1/q = 1$, the same proof gives us continuous bilinear functional on $C(\mathbf{R}^d) \otimes C^d(P)$. We cannot use Proposition 6 anymore, but using Schwartz's kernel theorem, we can (only) extend it to a distribution from $\mathcal{D}'(\mathbf{R}^d \times P)$. Therefore, our variant of the compensated compactness is confined on $L^p - L^q$ framework for $1/p + 1/q < 1$. However, under additional assumptions, we are able to prove the result in the optimal case $1/p + 1/q = 1$ (Corollary 15).

Before we proceed, let us recall the definition of fractional derivatives. For $\alpha \in \mathbf{R}^+$, we define $\partial_{x_k}^\alpha$ to be a pseudodifferential operator with a polyhomogeneous symbol $(2\pi i \xi_k)^\alpha$, i.e.

$$\partial_{x_k}^\alpha u = ((2\pi i \xi_k)^\alpha \hat{u}(\xi))^\vee.$$

In the sequel, we shall assume that sequences (\mathbf{u}_r) and (\mathbf{v}_r) are uniformly compactly supported. This assumption can be removed if the orders of derivatives $(\alpha_1, \dots, \alpha_d)$ are natural numbers. Otherwise, since the Leibnitz rule does not hold for fractional derivatives, the former assumption seems necessary.

Let us now introduce the localisation principle corresponding to an H -distribution.

Proposition 11. *Assume that sequences (\mathbf{u}_r) and (\mathbf{v}_r) are bounded in $L^p(\mathbf{R}^d; \mathbf{R}^N)$ and $L^q(\mathbf{R}^d; \mathbf{R}^N)$, where $1/p + 1/q < 1$, and converge toward $\mathbf{0}$ and $\mathbf{v} = (v_1, \dots, v_N)$ in the sense of distributions.*

Furthermore, assume that the sequence (\mathbf{u}_r) satisfies, for every $s = 1, \dots, M$:

$$G_{rs} := \sum_{j=1}^N \sum_{k=1}^d \partial_{x_k}^{\alpha_k} (a_{sjk} u_{jr}) \rightarrow 0 \text{ in } W^{-\alpha_1, \dots, -\alpha_d; p}(\mathbf{R}^d), \quad (19)$$

where $\alpha_k \in \mathbf{N}$ or $\alpha_k > d$, $k = 1, \dots, d$, and $a_{sjk} \in L^{\bar{s}'}(\mathbf{R}^d)$, $\bar{s} \in \langle 1, \frac{pq}{p+q} \rangle$.

Finally, by μ_{jm} denote the H -distribution (Theorem 9) corresponding to a pair of subsequences of (u_{jr}) and $(v_{mr} - v_m)$. Then the following relations hold in the sense of distributions for $m = 1, \dots, N$, $s = 1, \dots, M$ ($i = \sqrt{-1}$ below)

$$\sum_{j=1}^N \sum_{k=1}^n a_{sjk} (2\pi i \xi_k)^{\alpha_k} \mu_{jm} = 0. \quad (20)$$

Proof: Assume, without losing any generality, that $\mathbf{v} = \mathbf{0}$. Denote by \mathcal{B}_ψ the Fourier multiplier operator with the symbol

$$(\psi \circ \pi_P)(\boldsymbol{\xi}) \frac{(1 - \theta(\boldsymbol{\xi}))}{(|\xi_1|^{2\alpha_1} + \dots + |\xi_d|^{2\alpha_d})^{1/2}},$$

where θ is a cutoff function equal to one in a neighborhood of zero.

According to [10, Lemma 5], for any $\psi \in C^d(P)$ and any $\hat{s} > 1$, the multiplier operator $\mathcal{B}_\psi : L^2(\mathbf{R}^d) \cap L^{\hat{s}}(\mathbf{R}^d) \rightarrow W^{\alpha_1, \dots, \alpha_d; \hat{s}}(\mathbf{R}^d)$ is bounded (with $L^{\hat{s}}$ norm considered on the domain of \mathcal{B}_ψ); notice that the symbol of $\partial_{x_k}^{\alpha_k} \circ \mathcal{B}_\psi$ given by

$$(\psi \circ \pi_P)(\boldsymbol{\xi}) \frac{(1 - \theta(\boldsymbol{\xi}))(2\pi i \xi_k)^{\alpha_k}}{(|\xi_1|^{2\alpha_1} + \dots + |\xi_d|^{2\alpha_d})^{1/2}},$$

is a smooth, bounded function satisfying conditions of Marcinkiewicz's multiplier theorem ([17, Theorem IV.6.6'] or Corollary 2 here).

Insert in (19) the test function g_{rm} given by:

$$g_{rm}(\mathbf{x}) = \mathcal{B}_\psi(\phi v_{mr})(\mathbf{x}), \quad m \in \{1, \dots, N\} \quad (21)$$

where $\psi \in C^d(P)$ and $\phi \in C_c^\infty(\mathbf{R}^d)$. We get

$$\begin{aligned} \int_{\mathbf{R}^d} G_{rs} \overline{g_{rm}} d\mathbf{x} &= \int_{\mathbf{R}^d} \sum_{j=1}^N \sum_{k=1}^n a_{sjk} u_{jr} \overline{\mathcal{A}_{(\psi \circ \pi_P)(\boldsymbol{\xi}) \frac{(1 - \theta(\boldsymbol{\xi}))(2\pi i \xi_k)^{\alpha_k}}{(|\xi_1|^{2\alpha_1} + \dots + |\xi_d|^{2\alpha_d})^{1/2}}}(\phi v_{mr})} d\mathbf{x} \quad (22) \\ &= \int_{\mathbf{R}^d} \sum_{j=1}^N \sum_{k=1}^n a_{sjk} u_{jr} \overline{\mathcal{A}_{(\psi \circ \pi_P)(\boldsymbol{\xi}) \frac{(2\pi i \xi_k)^{\alpha_k}}{(|\xi_1|^{2\alpha_1} + \dots + |\xi_d|^{2\alpha_d})^{1/2}}}(\phi v_{mr})} d\mathbf{x} \\ &\quad - \int_{\mathbf{R}^d} \sum_{j=1}^N \sum_{k=1}^n a_{sjk} u_{jr} \overline{\mathcal{A}_{(\psi \circ \pi_P)(\boldsymbol{\xi}) \frac{\theta(\boldsymbol{\xi})(2\pi i \xi_k)^{\alpha_k}}{(|\xi_1|^{2\alpha_1} + \dots + |\xi_d|^{2\alpha_d})^{1/2}}}(\phi v_{mr})} d\mathbf{x}. \end{aligned}$$

Due to the boundedness properties of operator \mathcal{B}_ψ mentioned above and the compact support of ϕ , the sequence (g_{rm}) is bounded in $W^{\alpha_1, \dots, \alpha_d; t}(\mathbf{R}^d)$ for $t \in [1, q]$. Letting $r \rightarrow \infty$ in (22), we get (20) after taking into account Theorem 9 and the strong convergence of (G_{rs}) . Note that the second summand in the above identity goes to 0 because of the compact support of the function θ . \square

Remark 12. In the case $1/p + 1/q = 1$, taking into account Remark 10 and coefficients a_{sjk} from the space $C_0(\mathbf{R}^d)$, we get the same result as in (20) for distributions μ_{jm} from $\mathcal{D}'(\mathbf{R}^d \times P)$.

We can now formulate conditions under which (4) holds. We call them the strong consistency conditions. They represent a generalization of the standard consistency conditions given above.

As before, let $\bar{s} \in \langle 1, \frac{pq}{p+q} \rangle$ be a fixed number for given $p, q > 1$. Introduce the set

$$\Lambda_{\mathcal{D}} = \left\{ \boldsymbol{\mu} = (\mu_1, \dots, \mu_N) \in L_{w*}^{\bar{s}}(\mathbf{R}^d; (C^d(\mathbf{P}))')^N : \right. \\ \left. \sum_{j=1}^N \sum_{k=1}^d (2\pi i \xi_k)^{\alpha_k} a_{sjk} \mu_j = 0, \quad s = 1, \dots, M \right\}, \quad (23)$$

where the given equality is understood in the sense of $L_{w*}^{\bar{s}}(\mathbf{R}^d; (C^d(\mathbf{P}))')$.

Let us assume that

$$\begin{aligned} &\text{coefficients of the bilinear form } q \text{ from (3)} \\ &\text{belong to the space } L^t(\mathbf{R}^d), \text{ where } t \geq \bar{s}'. \end{aligned} \quad (24)$$

Remark that since $\bar{s} \in \langle 1, \frac{pq}{p+q} \rangle$ and $t \geq \bar{s}'$, it also must be $1/t + 1/p + 1/q < 1$.

Definition 13. We say that the set $\Lambda_{\mathcal{D}}$, bilinear form q from (3) satisfying (24), and the matrix $\boldsymbol{\mu} = [\mu_{jm}]_{j,m=1,\dots,N}$, $\mu_{jm} \in L_{w*}^{\bar{s}}(\mathbf{R}^d; (C^d(\mathbf{P}))')$ satisfy the strong consistency condition if for every fixed $m \in \{1, \dots, N\}$, the N -tuple $(\mu_{1m}, \dots, \mu_{Nm})$ belongs to $\Lambda_{\mathcal{D}}$, and it holds

$$\sum_{j,m=1}^N \langle \phi q_{jm} \otimes 1, \mu_{jm} \rangle \geq 0, \quad \phi \in C_c^\infty(\mathbf{R}^d; \mathbf{R}_0^+). \quad (25)$$

Under the given strong consistency condition, we have the following theorem.

Theorem 14. Assume that sequences (\mathbf{u}_r) and (\mathbf{v}_r) are bounded in $L^p(\mathbf{R}^d; \mathbf{R}^N)$ and $L^q(\mathbf{R}^d; \mathbf{R}^N)$, where $1/p + 1/q < 1$, and converge toward \mathbf{u} and \mathbf{v} in the sense of distributions. Assume that (19) holds.

Assume that

$$q(\mathbf{x}; \mathbf{u}_r, \mathbf{v}_r) \rightharpoonup \omega \quad \text{in } \mathcal{D}'(\mathbf{R}^d)$$

for the bilinear form q from (3) satisfying (24).

If the set $\Lambda_{\mathcal{D}}$, the bilinear form (3), and the (matrix of) H -distributions $\boldsymbol{\mu}$ corresponding to the sequences $(\mathbf{u}_r - \mathbf{u})$ and $(\mathbf{v}_r - \mathbf{v})$ satisfy the strong consistency condition, then it holds

$$q(\mathbf{x}; \mathbf{u}, \mathbf{v}) \leq \omega \quad \text{in } \mathcal{D}'(\mathbf{R}^d). \quad (26)$$

If in (25) stands equality, then we have equality in (26) as well.

Proof: Let us abuse the notation by denoting $\mathbf{u}_r = \mathbf{u}_r - \mathbf{u} \rightharpoonup \mathbf{0}$ and $\mathbf{v}_r = \mathbf{v}_r - \mathbf{v} \rightharpoonup \mathbf{0}$ as $r \rightarrow \infty$.

Remark that, according to Theorem 9, for any non-negative $\phi \in \mathcal{D}(\mathbf{R}^d)$

$$\lim_{r \rightarrow \infty} \int_{\mathbf{R}^d} \sum_{j,m=1}^N q_{jm} u_{jr} v_{mr} \phi \, d\mathbf{x} = \langle \phi \sum_{j,m=1}^N q_{jm} \otimes 1, \mu_{jm} \rangle, \quad (27)$$

where μ_{jm} is a H -distribution corresponding to sequences $u_{jr}, v_{mr} \rightharpoonup 0$. Since, according to the localisation principle (20), for every fixed $m \in \{1, \dots, N\}$, the N -tuple $(\mu_{1m}, \dots, \mu_{Nm})$ belongs to $\Lambda_{\mathcal{D}}$, we conclude from the strong consistency condition that

$$\langle \phi \sum_{j,m=1}^N q_{jm} \otimes 1, \mu_{jm} \rangle \geq 0.$$

From here, (27), and the fact that (since q is bilinear)

$$q(\mathbf{x}; \mathbf{u}_r, \mathbf{v}_r) \rightharpoonup \omega - q(\mathbf{x}; \mathbf{u}, \mathbf{v}) \geq 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^d),$$

the statement of the theorem follows. \square

If we assume that the sequence (\mathbf{v}_n) is bounded in $L^{p'}(\mathbf{R}^d; \mathbf{R}^N)$ and additionally assume that it can be well approximated by the truncated sequence $(T_l(\mathbf{v}_n))$, $l \in \mathbf{N}$, we can state the optimal variant of the compensated compactness as follows.

Corollary 15. *Assume that*

- *sequences (\mathbf{u}_r) and (\mathbf{v}_r) are bounded in $L^p(\mathbf{R}^d; \mathbf{R}^N)$ and $L^{p'}(\mathbf{R}^d; \mathbf{R}^N)$, where $1/p + 1/p' = 1$, and converge toward \mathbf{u} and \mathbf{v} in the sense of distributions;*
- *for every $l \in \mathbf{N}$, the sequences $(T_l(\mathbf{v}_r))$ converge weakly in $L^{p'}(\mathbf{R}^d; \mathbf{R}^N)$ toward \mathbf{h}^l , where the truncation operator T_l from (14) is understood coordinatewise;*
- *there exists a vector valued function $\mathbf{V} \in L^{p'}(\mathbf{R}^d; \mathbf{R}^N)$ such that $|\mathbf{v}_r| \leq \mathbf{V}$ holds coordinatewise for every $r \in \mathbf{N}$;*
- *(19) holds with $a_{skl} \in C_0(\mathbf{R}^d)$ and $q_{jm} \in C(\mathbf{R}^d)$.*

Assume that

$$q(\mathbf{x}; \mathbf{u}_r, \mathbf{v}_r) \rightharpoonup \omega \quad \text{in } \mathcal{D}'(\mathbf{R}^d).$$

If for every $l \in \mathbf{N}$, the set $\Lambda_{\mathcal{D}}$, the bilinear form (3), and the (matrix of) H -distributions $\boldsymbol{\mu}_l$ corresponding to the sequences $(\mathbf{u}_r - \mathbf{u})$ and $(T_l(\mathbf{v}_r) - \mathbf{h}^l)_r$ satisfy the strong consistency condition, then it holds

$$q(\mathbf{x}; \mathbf{u}, \mathbf{v}) \leq \omega \quad \text{in } \mathcal{D}'(\mathbf{R}^d). \quad (28)$$

If in (25) stands equality, then we have equality in (26) as well.

Proof: For every $l \in \mathbf{N}$, notice that $(q(\mathbf{x}; \mathbf{u}_r, T_l(\mathbf{v}_r)))_r$ is bounded in $L^p(\mathbf{R}^d)$:

$$\begin{aligned} \int_{\mathbf{R}^d} |q(\mathbf{x}; \mathbf{u}_r, T_l(\mathbf{v}_r))|^p d\mathbf{x} &\leq N^{2(p-1)} \sum_{j,m=1}^N \int_{\mathbf{R}^d} |q_{jm}|^p |u_{jr}|^p |T_l(v_{mr})|^p d\mathbf{x} \\ &\leq C_{N,l,p} \max_{j,m} (\|q_{jm}\|_{L^\infty(K)}^p \|u_{jr}\|_{L^p(K)}^p), \end{aligned}$$

where $K \subseteq \mathbf{R}^d$ is a compact set (remember that sequences (\mathbf{u}_r) , (\mathbf{v}_r) are uniformly compactly supported). Therefore, the sequence $(q(\mathbf{x}; \mathbf{u}_r, T_l(\mathbf{v}_r)))$ (we remind that l is fixed) admits a weak limit in $L^p(\mathbf{R}^d)$ (and thus in $\mathcal{D}'(\mathbf{R}^d)$) along a subsequence. Using a diagonal procedure, we can extract a subsequence (not relabeled) such that for every $l \in \mathbf{N}$ it holds

$$q(\mathbf{x}; \mathbf{u}_r, T_l(\mathbf{v}_r)) \rightharpoonup \omega_l \quad \text{in } \mathcal{D}'(\mathbf{R}^d).$$

where ω_l is a weak limit of $(q(\mathbf{x}; \mathbf{u}_r, T_l(\mathbf{v}_r)))_r$. According to the assumptions of the corollary on the strong consistency conditions involving $\boldsymbol{\mu}_l$ and the sequences $(\mathbf{u}_r - \mathbf{u})$ and $(T_l(\mathbf{v}_r) - \mathbf{h}^l)_r$, and Theorem 14 (remark that $(T_l(\mathbf{v}_r))_r$ is bounded), it holds

$$q(\mathbf{x}; \mathbf{u}, \mathbf{h}^l) \leq \omega_l \quad \text{in } \mathcal{D}'(\mathbf{R}^d). \quad (29)$$

We will finish the corollary if we show that for every nonnegative function $\varphi \in C_c^\infty(\mathbf{R}^d)$ it holds $\int_{\mathbf{R}^d} (\omega - q(\mathbf{x}; \mathbf{u}, \mathbf{v})) \varphi d\mathbf{x} \geq 0$. It holds

$$\begin{aligned}
\int_{\mathbf{R}^d} (\omega - q(\mathbf{x}; \mathbf{u}, \mathbf{v})) \varphi d\mathbf{x} &= \int_{\mathbf{R}^d} (\omega - q(\mathbf{x}; \mathbf{u}_r, \mathbf{v}_r)) \varphi d\mathbf{x} \\
&+ \int_{\mathbf{R}^d} (q(\mathbf{x}; \mathbf{u}_r, \mathbf{v}_r) - q(\mathbf{x}; \mathbf{u}_r, T_l(\mathbf{v}_r))) \varphi d\mathbf{x} \\
&+ \int_{\mathbf{R}^d} (q(\mathbf{x}; \mathbf{u}_r, T_l(\mathbf{v}_r)) - \omega_l) \varphi d\mathbf{x} + \int_{\mathbf{R}^d} (\omega_l - q(\mathbf{x}; \mathbf{u}, \mathbf{h}^l)) \varphi d\mathbf{x} \\
&+ \int_{\mathbf{R}^d} (q(\mathbf{x}; \mathbf{u}, \mathbf{h}^l) - q(\mathbf{x}; \mathbf{u}, \mathbf{v})) \varphi d\mathbf{x}.
\end{aligned} \tag{30}$$

Since the left hand side of (30) does not depend on r and l , we can take $\limsup_{l \rightarrow \infty} \lim_{r \rightarrow \infty}$ there. The first summand on the right hand side of the expression goes to zero according to the assumptions of the corollary; the third summand goes to zero according to the definition of ω_l ; we have established in (29) that the fourth summand is nonnegative. Let us show that the second summand in (30) goes to zero:

$$\begin{aligned}
\left| \int_{\mathbf{R}^d} (q(\mathbf{x}; \mathbf{u}_r, \mathbf{v}_r) - q(\mathbf{x}; \mathbf{u}_r, T_l(\mathbf{v}_r))) \varphi d\mathbf{x} \right| &\leq \int_{\mathbf{R}^d} |\varphi \mathbf{Q} \mathbf{u}_r \cdot (\mathbf{v}_r - T_l(\mathbf{v}_r))| d\mathbf{x} \\
&\leq \|\mathbf{Q} \mathbf{u}_r\|_{L^p} \|\varphi (\mathbf{v}_r - T_l(\mathbf{v}_r))\|_{L^{p'}},
\end{aligned}$$

where we have used the Hölder inequality. Since $\mathbf{v}_r - T_l(\mathbf{v}_r) \rightarrow 0$ pointwise, according to the assumption $|\mathbf{v}_r| \leq \mathbf{V}$ and the Lebesgue dominated convergence theorem, we conclude that $\|\varphi (\mathbf{v}_r - T_l(\mathbf{v}_r))\|_{L^{p'}} \rightarrow 0$ as $l, r \rightarrow \infty$ (or as $l \rightarrow \infty$ uniformly with respect to r).

For the last summand, we will proceed in a similar manner. Let us notice that we can write

$$\begin{aligned}
q(\mathbf{x}; \mathbf{u}, \mathbf{h}^l) - q(\mathbf{x}; \mathbf{u}, \mathbf{v}) &= \mathbf{Q} \mathbf{u} \cdot (\mathbf{h}^l - \mathbf{v}) \\
&= \mathbf{Q} \mathbf{u} \cdot ((\mathbf{h}^l - T_l(\mathbf{v}_r)) + (T_l(\mathbf{v}_r) - \mathbf{v}_r) + (\mathbf{v}_r - \mathbf{v})).
\end{aligned}$$

The first and the last summand on the right hand side of the last expression will go to zero according to the assumptions of the corollary. Concerning the second summand, from the Lebesgue dominated convergence theorem as before, we conclude $\limsup_{l \rightarrow \infty} \lim_{r \rightarrow \infty} \|(T_l(\mathbf{v}_r) - \mathbf{v}_r) \varphi\|_{L^1(\mathbf{R}^d)} = 0$. This concludes the proof. \square

Remark 16. The condition concerning existence of the dominating function \mathbf{V} from the previous theorem might look superfluous. However, as the following example shows, we cannot avoid it. Indeed, consider the case $d = N = 1$, $a = a_{111} = 0$. Let

$$u_r(\mathbf{x}) = v_r(\mathbf{x}) = \begin{cases} r, & |x| < r^{-2} \\ 0, & |x| \geq r^{-2} \end{cases}.$$

Then, $\|u_r\|_2 = 2$ for all $r \in \mathbf{N}$. Clearly, $u_r = v_r \rightarrow 0$ weakly as $r \rightarrow \infty$, while $T_l(u_r) \rightarrow 0$ as $r \rightarrow \infty$ strongly in $L^2(\mathbf{R})$ for every $l \in \mathbf{N}$. Therefore, the H -distributions μ_l corresponding to the sequences (u_r) and $(T_l(v_r))$ are trivial: $\mu_l \equiv 0$. Thus, the strong consistency condition is satisfied with the equality sign, but $q(u_r, v_r) = u_r^2 \rightarrow 2\delta(\mathbf{x}) \neq 0 = q(0, 0)$.

We would like to thank to the referee for this example.

In a conclusion of the section, we would like to make a comment concerning a connection between the standard consistency condition and, at least at first sight stronger, the strong consistency condition. To this end, note that we can rewrite the consistency condition (2) in the following form (we shall omit the second order derivatives since they have no influence on the reasoning below):

$$\Lambda_{\mathcal{F}} = \left\{ \lambda : \mathbf{R}^d \times S^{d-1} \rightarrow \mathbf{R}^N : \sum_{j=1}^N \sum_{k=1}^{\nu} a_{sjk}(\mathbf{x}) \xi_k \lambda_j(\mathbf{x}, \xi) = 0, \quad s = 1, \dots, M \right\}$$

and

$$q(\mathbf{x}; \lambda(\mathbf{x}, \xi), \lambda(\mathbf{x}, \xi)) \geq 0 \quad \text{for all } \lambda \in \Lambda_{\mathcal{F}} \text{ and all } (\mathbf{x}, \xi) \in \mathbf{R}^d \times S^{d-1}.$$

Having such a representation of the consistency condition, it seems reasonable to ask whether $\Lambda_{\mathcal{D}}$ is a closure of $\Lambda_{\mathcal{F}}$ in the sense of distributions. If this is the case, the generalisation presented here holds under the standard consistency condition. At this moment, we do not have any answer to this question.

However, we shall present an example showing that our approach can be used.

3. APPLICATION

Let us consider the non-linear parabolic type equation

$$L(u) = \partial_t u - \sum_{k,l=1}^d \partial_{x_l x_k} (a_{kl}(t, \mathbf{x}) g(t, \mathbf{x}, u)) \quad (31)$$

on $\Omega = \langle 0, \infty \rangle \times V$, where V is an open subset of \mathbf{R}^d . We assume that

$$\begin{aligned} u &\in L^p(\Omega), \quad g(t, \mathbf{x}, u) \in L^q(\Omega), \quad 1 < p, q, \\ a_{kl} &\in L_{loc}^s(\Omega), \quad \text{where } 1/p + 1/q + 1/s < 1, \end{aligned}$$

and that the matrix function $\mathbf{A} = [a_{kl}]_{k,l=1,\dots,d}$ is strictly positive definite on Ω , i.e.

$$\mathbf{A}\xi \cdot \xi > 0, \quad \xi \in \mathbf{R}^d \setminus \{0\}, \quad \text{a.e. } (t, \mathbf{x}) \in \Omega.$$

Furthermore, assume that g is a Carathéodory function and non-decreasing with respect to the third variable.

The following theorem holds.

Theorem 17. *Assume that sequences*

- (u_r) and $g(\cdot, u_r)$ are such that $u_r, g(u_r) \in L^2(\mathbf{R}^+ \times \mathbf{R}^d)$ for every $r \in \mathbf{N}$;
- that they are bounded in $L^p(\mathbf{R}^+ \times \mathbf{R}^d)$, $p \in \langle 1, 2 \rangle$, and $L^q(\mathbf{R}^+ \times \mathbf{R}^d)$, $q > 2$, respectively, where $1/p + 1/q < 1$;
- $u_r \rightharpoonup u$ and, for some, $f \in W^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d)$, the sequence

$$L(u_r) = f_r \rightarrow f \quad \text{strongly in } W^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d).$$

Under the assumptions given above, it holds

$$L(u) = f \quad \text{in } \mathcal{D}'(\mathbf{R}^+ \times \mathbf{R}^d).$$

Proof: Let us first define all functions on $\mathbf{R} \times \mathbf{R}^d$ by extending them with 0 out of $\mathbf{R}^+ \times \mathbf{R}^d$. Denote by w a distributional limit of $g(\cdot, u_r)$ along not relabeled

subsequence. Our first step is to show that the product of u_r and $g(\cdot, u_r)$ converges to uw in the sense of distributions. To do that, denote

$$u_{1r} = u_r - u, \quad u_{2r} = g(\cdot, u_r) - w. \quad (32)$$

Note that the following sequence of equations is satisfied

$$\partial_t u_{1r} - \sum_{k,l=1}^d \partial_{x_l x_k} (a_{kl} u_{2r}) = f_r - f, \quad (33)$$

and that $f_r - f$ tends to zero strongly in $W^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d)$. Introduce

$$\Lambda_{\mathcal{D}} = \left\{ \mu = (\mu_1, \mu_2) \in L_{w*}^{s'}(\mathbf{R}^+ \times \mathbf{R}^d; C^{d+1}(\mathbf{P})')^2 : -2i\pi\xi_0\mu_1 + 4\pi^2 \sum_{k,l=1}^d \xi_k \xi_l a_{kl} \mu_2 = 0 \right\}, \quad (34)$$

and remark that, according to the localisation principle given in Proposition 11,

$$(\mu_{12}, \mu_{22}) \in \Lambda_{\mathcal{D}} \quad (35)$$

for H -distributions μ_{12} and μ_{22} , corresponding to sequences (ϕu_{1r}) and (ϕu_{2r}) , and (ϕu_{2r}) and (ϕu_{2r}) , respectively. Above, $\phi \in C_c^2(\mathbf{R}^+ \times \mathbf{R}^d)$ is fixed.

From the localisation principle, for $\psi \in C^{d+1}(\mathbf{P})$ (here and in the sequel, symbols are real functions) and $\varphi \in C_c^2(\mathbf{R}^d)$, it holds

$$i\langle -2\pi\xi_0\psi\varphi, \mu_{12} \rangle + \langle 4\pi^2 \sum_{k,l=1}^d \xi_k \xi_l a_{kl}(\cdot, \cdot)\psi\varphi, \mu_{22} \rangle = 0. \quad (36)$$

Remark that for any $\psi \in C^{d+1}(\mathbf{P})$ the function $f_\psi = \langle \psi, \mu_{j2} \rangle$ is in $L^{s'}(\mathbf{R}^+ \times \mathbf{R}^d)$, $j = 1, 2$. For the functions f_ψ , where ψ belongs to a dense countable subset E of $C^{d+1}(\mathbf{P})$ containing a dense subset of odd and even functions (which we may choose since $C^{d+1}(\mathbf{P})$ is separable and we can represent every function as a sum of even and odd functions $\psi(\xi) = \frac{1}{2}(\psi(\xi) + \psi(-\xi)) + \frac{1}{2}(\psi(\xi) - \psi(-\xi))$), and the functions a_{kl} , $k, l = 1, \dots, d$, denote by $D \subseteq \mathbf{R}^+ \times \mathbf{R}^d$ the set of their common Lebesgue points (which is of full measure).

Now, fix $(t_0, \mathbf{x}_0) \in D$. According to the Plancherel theorem, we get

$$\int \overline{\varphi v} \mathcal{A}_\psi(\varphi v) = \int \overline{\widehat{\varphi v}} \psi \widehat{\varphi v} \in \mathbf{R} \quad (37)$$

for all $v \in L^2(\mathbf{R}^+ \times \mathbf{R}^d)$, real bounded multipliers ψ , and $\varphi \in C_c^2(\mathbf{R}^d)$. From here we conclude that

$$\langle 4\pi^2 \sum_{k,l=1}^d \xi_k \xi_l a_{kl}(t_0, \mathbf{x}_0) \psi \varphi, \mu_{22} \rangle \in \mathbf{R} \quad (38)$$

for any real multiplier ψ . Indeed, for a scalar matrix $\mathbf{A}(t_0, \mathbf{x}_0)$, taking into account that $4\pi^2 \mathbf{A}(t_0, \mathbf{x}_0) \xi \cdot \xi \geq 0$, we notice that $4\pi^2 \mathbf{A}(t_0, \mathbf{x}_0) \xi \cdot \xi \psi \boxtimes \varphi$ is a real function in ξ (where φ is constant with respect to ξ). Insert symbol $4\pi^2(\mathbf{A}(t_0, \mathbf{x}_0) \xi \cdot \xi \psi / \rho_{\mathbf{P}}) \boxtimes \varphi$ and sequences $u_r = v_r = \phi u_{2r}$ into definition (13) of H -distributions where

$$\rho_{\mathbf{P}} = (\xi_0^2 + \sum_{j=1}^d \xi_j^4)^{1/2}.$$

Now, the claim follows once we notice that equation (37) gives us a limit of real numbers.

On the other hand, from Lemma 5, we conclude that for any odd ψ , the function

$$\langle 4\pi^2 \sum_{k,l=1}^d \xi_k \xi_l a_{kl}(t_0, \mathbf{x}_0) \psi \varphi, \mu_{22} \rangle \in i\mathbf{R}. \quad (39)$$

Thus, from (38) and (39), we conclude that for any odd function ψ it must be

$$\langle 4\pi^2 \sum_{k,l=1}^d \xi_k \xi_l a_{kl}(t_0, \mathbf{x}_0) \psi \varphi, \mu_{22} \rangle = 0. \quad (40)$$

Taking into account (40), assuming $\psi \in E$, and inserting $(t, \mathbf{x}) = (t_0, \mathbf{x}_0)$ into (36), we conclude that for all points from D , it holds

$$\langle -2\pi \xi_0 \psi, \mu_{12}(t_0, \mathbf{x}_0, \cdot) \rangle = 0. \quad (41)$$

Now, since $u_r \in L^2(\mathbf{R}^+ \times \mathbf{R}^d)$ for every $r \in \mathbf{N}$, we can test (33) by $\overline{\varphi \mathcal{A}_{(1-\theta)\psi_P/\rho_P}(\varphi u_{1r})}$ where θ is a compactly supported even smooth function equal to one in a neighborhood of zero. Then, we let $r \rightarrow \infty$ and use the Plancherel theorem to obtain a relation similar to (36) (remark that $\mathcal{A}_{(1-\theta)\psi_P/\rho_P}$ is a compact $L^p \rightarrow L^p$ operator for any $p > 1$):

$$\begin{aligned} \lim_{r \rightarrow \infty} \int_{\mathbf{R}^{d+1}} -2\pi i \frac{(1-\theta(\xi))\xi_0}{\rho_P(\xi)} \psi_P(\xi) \mathcal{F}(\varphi u_{1r}) \overline{\mathcal{F}(\varphi u_{1r})} d\xi \\ + \langle 4\pi^2 \sum_{k,l=1}^d \xi_k \xi_l a_{kl}(\cdot, \cdot) \psi \varphi, \mu_{12} \rangle = 0, \end{aligned} \quad (42)$$

where, as usual, $\psi_P = \psi \circ \rho_P$. Denote by

$$\begin{aligned} I_r(\psi_P) &= \int_{\mathbf{R}^{d+1}} -2\pi i \frac{(1-\theta(\xi))\xi_0}{\rho_P(\xi)} \psi_P(\xi) \mathcal{F}(\varphi u_{1r}) \overline{\mathcal{F}(\varphi u_{1r})} d\xi \\ &= \int_{\mathbf{R}^{d+1}} -2\pi i \frac{(1-\theta(\xi))\xi_0}{\rho_P(\xi)} \psi_P(\xi) |\mathcal{F}(\varphi u_{1r})|^2 d\xi. \end{aligned} \quad (43)$$

We shall prove that for every even ψ

$$I_r(\psi_P) = 0. \quad (44)$$

Clearly, for any real ψ , it holds (see (43))

$$I_r(\psi_P) \in i\mathbf{R}. \quad (45)$$

However, from Lemma 5, we conclude that for any even ψ , it holds

$$\begin{aligned} I_r(\psi_P) &= \int_{\mathbf{R}^+ \times \mathbf{R}^d} \varphi(\mathbf{x}) u_{1r}(t, \mathbf{x}) \partial_t \overline{(\mathcal{A}_{(1-\theta)\psi_P/\rho_P}(\varphi u_{1r}))(t, \mathbf{x})} dt d\mathbf{x} \\ &= \int_{\mathbf{R}^+ \times \mathbf{R}^d} \varphi(\mathbf{x}) u_{1r}(t, \mathbf{x}) \partial_t (\mathcal{A}_{(1-\theta)\psi_P/\rho_P}(\varphi u_{1r}))(t, \mathbf{x}) dt d\mathbf{x} \in \mathbf{R}. \end{aligned}$$

Being both purely real for any even ψ and purely imaginary for any ψ (see (45)), it follows that $I_r(\psi_P)$ must be zero for any even ψ . From here, (44) follows.

Now, since the function $\varphi \in C_c^2(\mathbf{R}^{d+1})$ is arbitrary, from (42) we get the following relation for every (Lebesgue) point $(t, \mathbf{x}) \in D$ and $\psi \in E$:

$$\langle 4\pi^2 \sum_{k,l=1}^d \xi_k \xi_l a_{kl}(t, \mathbf{x}) \psi_2, \mu_{12}(t, \mathbf{x}, \cdot) \rangle = 0. \quad (46)$$

Since the set D is of full measure, summing the results from (41) and (46), we conclude that for any odd symbol $\psi_1 \in E$ and even symbol $\psi_2 \in E$, we have

$$\langle 2\pi \xi_0 \psi_1 \varphi, \mu_{12} \rangle + \langle 4\pi^2 \sum_{k,l=1}^d \xi_k \xi_l a_{kl}(t, \mathbf{x}) \psi_2 \varphi, \mu_{12} \rangle = 0.$$

Thus, by taking $\psi_1 = \xi_0 \psi$ and $\psi_2 = \psi$ for an even symbol $\psi \in E$, we conclude:

$$\left\langle \left(2\pi \xi_0^2 + 4\pi^2 \sum_{k,l=1}^d \xi_k \xi_l a_{kl}(t, \mathbf{x}) \right) \psi \varphi, \mu_{12} \right\rangle = 0. \quad (47)$$

Since μ_{12} is continuous on $L^s(\mathbf{R}^{d+1}; C^{d+1}(\mathbf{P}))$, we conclude that (47) holds for any even $\psi \in C^{d+1}(\mathbf{P})$.

Since the function

$$f(t, \mathbf{x}, \boldsymbol{\xi}) = \frac{\varphi}{2\pi \xi_0^2 + 4\pi^2 \sum_{k,l=1}^d \xi_k \xi_l a_{kl}} \in L^s(\mathbf{R}^+ \times \mathbf{R}^d; C^{d+1}(\mathbf{P}))$$

is even with respect to the variable $\boldsymbol{\xi}$, we conclude from (47) (we can put f instead $\varphi\psi$ there) that

$$\langle 1 \otimes \varphi, \mu_{12} \rangle = 0. \quad (48)$$

From (35) and (48), we conclude that the following bilinear form

$$q(\mathbf{x}; \boldsymbol{\lambda}, \boldsymbol{\eta}) = \lambda_1 \eta_2, \quad \boldsymbol{\lambda} = (\lambda_1, \lambda_2), \quad \boldsymbol{\eta} = (\eta_1, \eta_2),$$

satisfies the strong consistency condition with the set $\Lambda_{\mathcal{D}}$ introduced in (34). Now we can apply Theorem 14 to conclude that

$$q(\mathbf{x}; (u_{1r}, u_{2r}), (u_{2r}, u_{2r})) = u_{1r} u_{2r} \rightharpoonup 0 = q(\mathbf{x}; (0, 0), (0, 0)) \text{ in } \mathcal{D}'(\mathbf{R}^+ \times \mathbf{R}^d) \quad (49)$$

since both $u_{1r} = u_r - u$ and $u_{2r} = g(\cdot, u_r) - w$ weakly converge to 0. Using the bilinearity of q , we conclude

$$u_r g(\cdot, u_r) \rightharpoonup uw \text{ in } \mathcal{D}'(\mathbf{R}^+ \times \mathbf{R}^d). \quad (50)$$

Our next step is to identify $g(\cdot, u)$ as a weak limit of $g(\cdot, u_r)$. To do that we will employ the theory of Young measures. Up to this moment we didn't need any assumption on the function g itself, only on the sequence $g(\cdot, u_r)$.

Denote by $\eta_{t,\mathbf{x}}$ the Young measure associated to a subsequence of the sequence (u_r) . Since g is a Carathéodory function, from (32) and (50), it holds [16]:

$$\begin{cases} u(t, \mathbf{x}) = \int \lambda d\eta_{t,\mathbf{x}}(\lambda), \\ w(t, \mathbf{x}) = \int g(t, \mathbf{x}, \lambda) d\eta_{t,\mathbf{x}}(\lambda), \end{cases} \quad (51)$$

and

$$u(t, \mathbf{x}) \int g(t, \mathbf{x}, \lambda) d\eta_{t,\mathbf{x}}(\lambda) = u(t, \mathbf{x}) w(t, \mathbf{x}) = \int \lambda g(t, \mathbf{x}, \lambda) d\eta_{t,\mathbf{x}}(\lambda).$$

The latter equality implies

$$\begin{aligned} \int (\lambda - u(t, \mathbf{x}))g(t, \mathbf{x}, \lambda)d\eta_{t,\mathbf{x}}(\lambda) = \\ \int \left(\lambda - u(t, \mathbf{x}) \right) \left(g(t, \mathbf{x}, \lambda) - g(t, \mathbf{x}, u(t, \mathbf{x})) \right) d\eta_{t,\mathbf{x}}(\lambda) = 0, \end{aligned} \quad (52)$$

because

$$\begin{aligned} \int (\lambda - u)g(t, \mathbf{x}, u)d\eta_{t,\mathbf{x}}(\lambda) &= g(t, \mathbf{x}, u) \int \lambda d\eta_{t,\mathbf{x}}(\lambda) - g(t, \mathbf{x}, u)u \int d\eta_{t,\mathbf{x}}(\lambda) \\ &= g(t, \mathbf{x}, u)u - g(t, \mathbf{x}, u)u \\ &= 0, \end{aligned}$$

where function u does not depend on λ and we have used first equality in (51) and the fact that $\eta_{t,\mathbf{x}}$ is a probability measure.

Since g is non-decreasing with respect to λ , we conclude from (52)

$$g(t, \mathbf{x}, \lambda) = g(t, \mathbf{x}, u(t, \mathbf{x})) \quad \text{on } \text{supp}\eta_{t,\mathbf{x}},$$

which implies

$$w(t, \mathbf{x}) = \int g(t, \mathbf{x}, \lambda)d\eta_{t,\mathbf{x}}(t, \mathbf{x}) = g(t, \mathbf{x}, u(t, \mathbf{x})).$$

From here, we finally conclude that

$$L(u_r) \rightharpoonup L(u) = f \quad \text{in } \mathcal{D}'(\mathbf{R}^+ \times \mathbf{R}^d).$$

□

4. ACKNOWLEDGEMENTS

The authors would like to thank to prof. Nenad Antić and to the unknown referee for their generous help during the work on the article.

The research is supported by the bilateral project Transport in highly heterogeneous media between Croatia and Montenegro, by the Croatian Science Foundation's funding of the project "Weak convergence methods and applications" number 9780, and by the project "Advection-diffusion equations in highly heterogeneous media" of the Montenegrin Ministry of Science. Part of the work was carried out while M.Mišur was visiting the Basque Center for Applied Mathematics, within the frame of FP7-246775 NUMERIWAVES project of the ERC.

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